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14. ABSTRACT

The study illustrates the effectiveness of three-phase functionally graded material systems where stiff (typically ceramic) particles are incorporated within the fiber-reinforced medium. Added particles increase the stiffness of the fiber-reinforced material reducing static and dynamic stresses and deformations, increasing buckling loads and fundamental frequencies and enhancing the response to blast loading.

The micromechanical model presented in the study represents an extrapolation of the Mori-Tanaka homogenization approach to the case of the system with dissimilar inclusions. It is shown that the material constants (elastic moduli and

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Report Title

Spatially tailored and functionally graded light-weight structures for optimum mechanical performance

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The micromechanical model presented in the study represents an extrapolation of the Mori-Tanaka homogenization approach to the case of the system with dissimilar inclusions. It is shown that the material constants (elastic moduli and Poisson's ratios) obtained by the method developed in the study are within the Voight-Reuss and Hashin-Shtrikman bounds. Moreover, the stiffness of a representative cross-ply material remains within the strict three-point bounds. Accordingly, the developed micromechanical model is applicable to the analysis of cross-ply functionally graded particulate-matrix fiber-reinforced materials.

The tensors of stiffness obtained by the model are applied to illustrate the effectivness of three-phase particulate-matrix fiber-reinforced materials in representative static and dynamic problems. In particular, it is shown that blast-induced deformations and stresses can be significantly reduced by adding only a small amount of particles to the outer layers of a fiber-reinforced material.

List of papers submitted or published that acknowledge ARO support during this reporting period. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

V. Birman and G. Genin, "Micromechanics and Response of Particulate-Matrix Fiber-Reinforced Functionally Graded Composites"

This paper will be publsihed in the special issue of the International Journal of Solids and Structures dedicated to the memory of Professor Liviu Librescu. The issue is planned for publication in 2009. PI (Vicor Birman) is Guest Editor of the issue.

Number of Papers published in peer-reviewed journals: 1.00

(b) Papers published in non-peer-reviewed journals or in conference proceedings (N/A for none)

Number of Papers published in non peer-reviewed journals: 0.00

(c) Presentations

The paper based on the present research will be presented at IMECE 2008 (International Mechanical Engineering Congress and Exposition), at the Symposium in Memory of Professor Liviu Librescu.

Number of Presentations: 1.00

Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts): 0

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See publsihed peer-reviewed paper.

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Number of Inventions:

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NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Names of Post Doctorates

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Names of Faculty Supported

NAME	PERCENT_SUPPORTED	National Academy Member
Victor Birman	1.00	No
FTE Equivalent:	**1.00**	
Total Number:	**1**	

Names of Under Graduate students supported

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Student Metrics

This section only applies to graduating undergraduates supported by this agreement in this reporting period

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Names of Personnel receiving masters degrees

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Total Number:

Names of personnel receiving PHDs

NAME

Total Number:

Names of other research staff

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Sub Contractors (DD882)

Inventions (DD882)

The following attachment represents the abbreviated version of the paper for the special issue of the International Journal of Solids and Structures honoring Professor Liviu Librescu. The numerical section is limited by the micromechanical analysis (additional results for static pressure and blast loading will be included in the complete version of the paper)

Micromechanics and Response of Particulate-Matrix Fiber-Reinforced Functionally Graded Composites

Abstract

Reinforcement of fibrous composites by stiff particles embedded in the matrix offers the potential for simple, economic functional grading, enhanced response to mechanical loads and improved functioning at high temperature. The solution shown in the paper extends a version of the Mori-Tanaka micromechanics to the case of a fiber-reinforced material with inclusions embedded within the matrix. Furthermore, we present several bounds and estimates for the mechanical properties of such composite and show how functional grading within these bounds can affect the performance of a structure. The representative example presented for a rectangular simply supported panel subject to uniform transverse pressure illustrates an abrupt decrease in deformations due to the addition of a modest amount of particles. The other example suggests superior blast resistance of the panel achieved at the expense of only a small increase in weight.

1. Introduction

Hybrid composite materials consisting of an isotropic polymer matrix reinforced by both particles and unidirectional fibers offer the potential for simple functional grading to tailor mechanical response and reduce stress concentrations around attachments and discontinuities. Local particle reinforcement can increase stiffness and strength at key locations at the expense of a relatively small increase in weight. Moreover, polymeric composite structures subject to thermal loading exhibit matrix deterioration at high temperature, but a matrix reinforced with ceramic particles offers the potential to increase the structural endurance and load capacity at high temperature (Birman and Byrd, 2007).

The application of hybrid particulate-matrix fiber-reinforced composites involves an analysis of the stiffness of such materials that can be applied to structures with either uniform or variable property distribution. For example, the latter structures may represent thin-walled shells or plates with in-surface variable stiffness. A particularly simple and attractive grading scheme involves embedding particles only in the outer layers of a laminate, achieving maximal increases in bending stiffness with a minimum of additional weight. Tailoring of the volume fraction of particles, especially on the outer laminae of a composite, is far simpler than the alternative of varying fiber volume fraction or their orientation.

A broad range of homogenization methods exist to predict the properties of composite materials including particulate inclusions. Several recent reviews and articles outlining these methods are available (e.g., Tucker and Liang, 1999 and Kakavas and Kontoni, 2006). Additionally, the study of Hu and Weng (2000) is of interest as it outlines and compares micromechanical models, including the double-inclusion method (Hori and Nemat-Nasser 1993), and the models of Ponte Castaneda and Willis (1995) and of Kuster-Toksoz (1974) and the well known Mori-Tanaka model (1973).

A general approach to the characterization of a hybrid composite consisting of three different phases was proposed by Kanaun and Jeulin (2001), based on the effective field method. This method is based on the assumption that the strain field acting on every inclusion varies for different populations. The disadvantage of the method is related to its high dependency on the precision of the micromechanical information about constituent phases, so that while the theoretical solution may be accurate, practical implementations would depend on the tight control of the manufacturing process that cannot be achieved at a reasonable cost in industry.

The analysis presented here is based on the use of the modification of the Mori-Tanaka method (Mori and Tanaka, 1973) suggested by Benveniste (1987). In addition, the upper and lower bounds of the material constants (moduli of elasticity, and shear, the bulk modulus of a particulate material and the Poisson ratios) are determined using the bounding techniques. Available bounds include those by Voight and Reuss (Hill, 1952), the Hashin method for a particulate material (Hashin, 1962) combined with the Hashin-Shtrikman bounds for a fiber-reinforced material (Hashin and Shtrikman, 1962, 1963), the Weng approach (Weng, 1992) and the three-point bounds technique (Milton and Phan-Tsien, 1982). The accuracy of these methods has been debated in literature. For example, the modulus of elasticity of a particulate material can be accurately evaluated using the Mori-Tanaka method only if the volume fraction of the inclusions does not exceed 40% (Kwon and Dharan, 1995; Sun et al., 2007). The Hashin bounds are also imperfect predicting an exceedingly wide spectrum of the modulus of elasticity at large particle volume fractions. At the same time, the predictions obtained by the Mori-Tanaka and Hashin methods for the Poisson ratio of particulate composites were found quite accurate (Sun et al., 2007). On the other hand, Noor and Shah (1993) showed that the Mori-Tanaka method provides an accurate prediction of the properties of fiber-reinforced composites even at a high volume fraction of fibers. In the following analysis the volume fraction of particles is assumed quite small reflecting the presence of fibers that occupy a significant volume within the material. Accordingly, it is acceptable to rely on the Mori-Tanaka method while the bounding techniques are also applied to reflect a range of anticipated variations of the materials properties.

The paper presents a formulation of micromechanics for a particulate-matrix fiber-reinforced material based on the extrapolation of the Benveniste approach (1987) for the case of dissimilar inclusions (fibers and particles). Subsequently, the bounding techniques outlined above are applied in a two-step solution: the bounds are first established for a particulate matrix and subsequently, these bounds are employed to establish bounds for this matrix reinforced by unidirectional fibers. Furthermore, the results of the homogenation conducted by on the particulate-matrix laminae are applied to predict the properties of a cross-ply material consisting of such laminae. Finally, the advantages available embedding a small amount of particles in fiber-reinforced materials are illustrated on the examples of a simply supported cross-ply composite plate subject to static pressure or dynamic blast overpressure. In both these examples, adding glass particles to a glass/epoxy plate material resulted in significant reductions of maximum deformations achieved with a very modest additional weight.

2. Benveniste-Type Estimate of the Stiffness of a Fiber-Reinforced Particulate Material

Consider a material where two different types of isotropic inclusions are distributed within an isotropic matrix. The properties of the matrix are identified in the subsequent solution with the subscript $i = 1$, while two types of the inclusions are denoted by $i = 2$ and $i = 3$. We take phase 2 to be spherical particles and phase 3 to be aligned fibers. Each type of inclusion possesses a definite geometry. Then the stiffness tensor of the material can be derived as a generalization of the Mori-Tanaka approach. In the present paper this generalization is based on the solution by Benveniste (1987) for a particulate composite with a single type of inclusions.

The approach is based on the following assumptions:

1. All material phases are isotropic and linearly-elastic.
2. The perturbed strain in the matrix due to the presence of inclusions is not affected by the interaction of two types of inclusions. In other words, each type of inclusion i = 2 and i = 3 affect the strains in the matrix, but the perturbed matrix strain due to the interaction between these inclusions that is assumed to be of the second order is neglected.
3. Phase 3 is represented by unidirectionally-oriented fibers of circular cross section; i.e., the composite material is a lamina with embedded particles. This assumption is only needed to utilize the Eshelby tensor for cylindrical inclusions. In general, the derivation shown below is independent of the orientation and shape of fibers and particles as long as the corresponding Eshelby tensor is known.

The average stress and average strain tensors for the material under consideration are related through the effective stiffness tensor:

$$\bar{\sigma} = \overline{\mathbf{L}}\bar{\varepsilon} \tag{1}$$

where the bar denotes a volumetric average.

The effective stiffness tensor for a matrix with two different types of embedded inclusions can be given as a generalization of the expression proposed by Hill (1963):

$$\overline{\mathbf{L}} = \overline{\mathbf{L}}_1 + f_2\left(\overline{\mathbf{L}}_2 - \overline{\mathbf{L}}_1\right)\overline{\mathbf{A}}_2 + f_3\left(\overline{\mathbf{L}}_3 - \overline{\mathbf{L}}_1\right)\overline{\mathbf{A}}_3 \tag{2}$$

where $\overline{\mathbf{L}}_i$ is the stiffness tensor of the i^{th} phase, f_2 and f_3 are volume fractions of the corresponding types of inclusions, while the tensor of concentration factors $\overline{\mathbf{A}}_2$ and $\overline{\mathbf{A}}_3$ represent the relationship between the tensors of average strains in the corresponding inclusions $(\bar{\varepsilon}_i)$ and the mean remote strain tensor $(\bar{\varepsilon}_0)$:

$$\bar{\varepsilon}_2 = \overline{\mathbf{A}}_2\bar{\varepsilon}_0, \qquad \bar{\varepsilon}_3 = \overline{\mathbf{A}}_3\bar{\varepsilon}_0 \tag{3}$$

Note according to the second assumption, this approach does not explicitly account for the interaction of different types of inclusions. Accordingly, it is applicable only if at least one type of inclusion has a relatively small volume fraction.

Expanding the Mori-Tanaka (1973) ideas, the tensors of average strain in the matrix and in inclusions are represented as

$$
\begin{aligned}
\bar{\varepsilon}_1 &= \bar{\varepsilon}_0 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 \\
\bar{\varepsilon}_2 &= \bar{\varepsilon}_0 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 + \varepsilon_2' \\
\bar{\varepsilon}_3 &= \bar{\varepsilon}_0 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 + \varepsilon_3'
\end{aligned}
\tag{4}
$$

where $\tilde{\varepsilon}_i$ are tensors of perturbations superimposed on the average strain in the matrix as a result of the presence of the corresponding inclusions, and ε_i' are tensors of average perturbed strain in the inclusions relative to the tensor of average strain in the matrix.

The tensors of average stresses in the inclusions can now be expressed in terms of the stiffness of the matrix:

$$
\begin{aligned}
\overline{\mathbf{L}}_2\left(\bar{\varepsilon}_0 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 + \varepsilon_2'\right) &= \overline{\mathbf{L}}_1\left(\bar{\varepsilon}_0 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 + \varepsilon_2' - \varepsilon_2^*\right) \\
\overline{\mathbf{L}}_3\left(\bar{\varepsilon}_0 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 + \varepsilon_3'\right) &= \overline{\mathbf{L}}_1\left(\bar{\varepsilon}_0 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 + \varepsilon_3' - \varepsilon_3^*\right)
\end{aligned}
\tag{5}
$$

where ε_i^* are tensors of average correlation strain in the corresponding type of inclusions. These tensors are related to the tensors of perturbations in the inclusions defined above by

$$
\varepsilon_2^* = \overline{\mathbf{S}}_2^{\,-1}\varepsilon_2' \qquad\qquad \varepsilon_3^* = \overline{\mathbf{S}}_3^{\,-1}\varepsilon_3' \tag{6}
$$

where $\overline{\mathbf{S}}_i^{-1}$ are fourth-order Eshelby's tensors. These tensors are presented in the Appendix 1 for the cases of spherical inclusions and for infinitely long cylindrical inclusions (fibers).

It is evident from (4) and (5) that the tensors of perturbation strains can be expressed in terms of the tensors of average strain in the corresponding inclusions as

$$
\varepsilon_i' = \overline{\mathbf{S}}_i\overline{\mathbf{L}}_1^{-1}\left(\overline{\mathbf{L}}_1 - \overline{\mathbf{L}}_i\right)\bar{\varepsilon}_i \qquad\qquad (i = 2,3). \tag{7}
$$

As also directly follows from (4),

$$\varepsilon_i' = \bar{\varepsilon}_i - \bar{\varepsilon}_1 . \tag{8}$$

It will be necessary for the subsequent transformation to express the tensors of average strain in each type of inclusions in terms of the tensor of the average strain in the matrix, i.e. to find the coefficient tensors in the equations

$$\bar{\varepsilon}_i = \overline{\mathbf{T}}_i \bar{\varepsilon}_1 \tag{9}$$

This is easily accomplished using (7) and (8):

$$\overline{\mathbf{T}}_i = \left[\bar{\mathbf{I}} + \overline{\mathbf{S}}_i \overline{\mathbf{L}}_1^{-1} \left(\overline{\mathbf{L}}_i - \overline{\mathbf{L}}_1 \right) \right]^{-1} \tag{10}$$

where $\bar{\mathbf{I}}$ is a fourth-order unit tensor.

The tensors of concentration factors are now determined expressing the tensor of the applied strain that also represents the average strain in an equivalent homogeneous material in terms of strains in the constituent phases through the rule of mixtures:

$$\bar{\varepsilon}_0 = f_1 \bar{\varepsilon}_1 + f_2 \bar{\varepsilon}_2 + f_3 \bar{\varepsilon}_3 \tag{11}$$

Using eqns. (3) and (9) in (11) yields

$$\overline{\mathbf{A}}_i = \overline{\mathbf{T}}_i \left(f_1 \bar{\mathbf{I}} + f_2 \overline{\mathbf{T}}_2 + f_3 \overline{\mathbf{T}}_3 \right)^{-1} \tag{12}$$

Using the concentration tensors as given by (12) in the tensor of effective stiffness (2) yields the solution:

$$\overline{\mathbf{L}} = \overline{\mathbf{L}}_1 + \sum_{i=2}^{3} f_i \left(\overline{\mathbf{L}}_i - \overline{\mathbf{L}}_1 \right) \overline{\mathbf{T}}_i \left(f_1 \bar{\mathbf{I}} + f_2 \overline{\mathbf{T}}_2 + f_3 \overline{\mathbf{T}}_3 \right)^{-1} \tag{13}$$

The computations utilizing (13) can be performed using the tensor decomposition presented by Walpole (1983) and summarized recently by Sevostianov and Kachanov (2007).

Note that in case of a single type of inclusions this result converges to the formula derived by Benveniste (1987).

3. Two-Step Approach to the Evaluation of the Elastic Response of a Particulate-Matrix Fiber-Reinforced Composite Material

According to the approach adopted in this paper, conventional bounds and estimates are applied to evaluate the tensor of stiffness of the isotropic particle-reinforced matrix. Subsequently, this tensor of stiffness is used to obtain bounds and estimates for the stiffness of a composite comprised of isotropic fibers embedded in this isotropic particle-reinforced matrix. In this section, we concentrate on three-point bounds since as shown in numerical examples they appear provide the tightest range of possible values of material constants.

3.1 Bounds and Estimates of the Mechanical Response of the Particle Reinforced Matrix

Beran and Molyneux (1966) and McCoy (1970) obtained three-point bounds on the effective bulk modulus and the effective shear modulus for two-phase composites. These contain two calculated parameters, ζ_2 and η_2 (additionally, $\eta_1 = 1 - \eta_2$, $\zeta_1 = 1 - \zeta_2$) that characterize the shape and distribution of the two phases (e.g. Torquato, 1991) and must be evaluated numerically. Milton and Phan-Thien (1982) further improved the McCoy (1970) shear-modulus bounds.

The three-point Milton-Phan-Thien (1982) bounds on the effective shear modulus G_e for isotropic two-phase composites are given by:

$$\langle G \rangle - \frac{\phi_1 \phi_2 (G_2 - G_1)^2}{\langle \tilde{G} \rangle + \Xi} \le G_e \le \langle G \rangle - \frac{\phi_1 \phi_2 (G_2 - G_1)^2}{\langle \tilde{G} \rangle + \Theta} \tag{14}$$

where

$$\Xi = \frac{\left\langle \frac{128}{K} + \frac{99}{G} \right\rangle_\zeta + 45 \left\langle \frac{1}{G} \right\rangle_\eta}{30 \left\langle \frac{1}{G} \right\rangle_\zeta \left\langle \frac{6}{K} - \frac{1}{G} \right\rangle_\zeta + 6 \left\langle \frac{1}{G} \right\rangle_\eta \left\langle \frac{2}{K} + \frac{21}{G} \right\rangle_\zeta}, \text{ and} \tag{15}$$

$$\Theta = \frac{3\langle G\rangle_\eta \langle 6K + 7G\rangle_\zeta - 5\langle G\rangle_\zeta^2}{6\langle 2K - G\rangle_\zeta + 30\langle G\rangle_\eta}, \tag{16}$$

in which $\langle\ \rangle$ denotes a weighted average ($\langle G\rangle = G_1\phi_1 + G_2\phi_2$, $\langle G\rangle_\zeta = G_1\zeta_1 + G_2\zeta_2$, $\langle G\rangle_\eta = G_1\eta_1 + G_2\eta_2$), and a tilde represents a reverse-weighted average, e.g. $\langle\tilde{G}\rangle = G_2\phi_1 + G_1\phi_2$. Here, ϕ_1 and ϕ_2 are the volume fractions of epoxy and (relatively stiff) spherical particles, respectively, within the particulate-reinforced matrix; these relate to f_1 and f_2 as $\phi_2 = f_2 /(f_1 + f_2)$, with $\phi_1 = 1 - \phi_2$.

The simplified form (Milton, 1981) of the three-point Beran-Molyneux (1966) bounds on the bulk modulus K_e of isotropic two-phase composites is given by:

$$\langle K\rangle - \frac{\phi_1\phi_2(K_2 - K_1)^2}{\langle\tilde{K}\rangle + \frac{2(d-1)}{d}\langle G^{-1}\rangle_\zeta^{-1}} \le K_{pm}^{eff} \le \langle K\rangle - \frac{\phi_1\phi_2(K_2 - K_1)^2}{\langle\tilde{K}\rangle + \frac{2(d-1)}{d}\langle G\rangle_\zeta} \tag{17}$$

where $d = 3$.

Calculation of ζ_2 and η_2 is computationally expensive, and values have been reported for only a limited number of microstructures at this time. These values range between $0.15\phi_2<\zeta_2<\phi_2$ and $0.5\phi_2<\eta_2<\phi_2$ (Torquato, 1991). For randomly-spaced spherical particles, ζ_2~$0.211\phi_2$ and η_2~$0.483\phi_2$ (Torquato, 2001). The dense packing limit for spherical particles is that ϕ_2 cannot exceed approximately 0.63.

Approximations lying between these bounds are available:

$$\phi_2 \frac{\kappa_{21}}{\kappa_{e1}} = 1 - \frac{(d+2)(d-1)G_1\kappa_{21}\mu_{21}}{d(K_1 + 2G_1)}\phi_1\zeta_2 \tag{18}$$

and

$$\phi_2 \frac{\mu_{21}}{\mu_{e1}} = 1 - \frac{2G_1\kappa_{21}\mu_{21}}{d(K_1 + 2G_1)}\phi_1\zeta_2 - \frac{\zeta_2\left(d^2 - 4\right)G_1(2K_1 + 3G_1) + \eta_2\left(dK_1 + (d-2)G_1\right)^2}{2d(K_1 + 2G_1)^2}\mu_{21}^2\phi_1, \tag{19}$$

where d=3, $\kappa_{21}=\dfrac{K_2-K_1}{K_2-AG_1}$, $\kappa_{e1}=\dfrac{K_e-K_1}{K_e-AG_1}$, $\mu_{21}=\dfrac{G_2-G_1}{G_2-BG_1}$, $\mu_{e1}=\dfrac{G_e-G_1}{G_e-BG_1}$, $A=\dfrac{2(d-1)}{d}$, and

$$B=\frac{dK_1/2_1+(d+1)(d-2)G_1/d}{K_1+2G_1}.$$

3.2 Bounds and Estimates of the Mechanical Response of the Fiber- and Particle-Reinforced Lamina

We now study the mechanics of a lamina containing aligned fibers of volume fraction f_3 embedded in an isotropic matrix of volume fraction $f_1 + f_2$ with the above characteristics. The dense packing limit requires $f_3 < 0.83$. Five separate moduli are needed to fully describe the linear elastic response of this transversely isotropic material but only three of the effective moduli are independent (Hill, 1963).

Transverse shear modulus

The Silnutzer three-point lower bound on the transverse shear modulus, G_e^T, is given by Eq. (14) and Eq. (15) (Silnutzer, 1972; Torquato, 2001) with ϕ_1 replaced with $(f_1 + f_2)$, ϕ_2 replaced with f_3, $\eta_2 = 0.276\, f_3$, $\zeta_2 = 0.691\, f_3 + 0.0428\ \ f_3^2$, and $\langle\ \rangle$ replaced with $\langle\ \rangle_f$, where, for example, $\langle G\rangle_f \equiv (f_1+f_2)G_{pm} + f_3G_{fib}$. Here, a subscript of *fib* refers to properties of the fibers (i=3), a subscript of *pm* refers to properties of the particle reinforced matrix (i=1 and 2).

The Gibiansky-Torquato (1995) upper bound for G_e^T is tighter than the Silnutzer bound. Making the above substitutions, the Gibiansky-Torquato upper bound is obtained from Eq. (14) using the following definition for Θ:

$$\Theta^{-1}=\begin{cases}\left\langle\dfrac{1}{G^{-1}+K_{\max}^{-1}}\right\rangle_\eta^{-1}+\dfrac{1}{K_{\max}}, & t\le-\dfrac{1}{K_{\max}}\\[2ex] 2\left\langle K^{-1}\right\rangle_\zeta+\left\langle G^{-1}\right\rangle_\eta-\dfrac{[H+Z]^2}{\left\langle\tilde{G}^{-1}\right\rangle_\eta+2\left\langle\tilde{K}^{-1}\right\rangle_\zeta}, & -\dfrac{1}{K_{\max}}\le t\le\dfrac{1}{G_{fib}}\\[2ex] 2\left\langle\dfrac{1}{G_{fib}^{-1}+K^{-1}}\right\rangle_\eta^{-1}+\dfrac{1}{G_{fib}}, & t\ge\dfrac{1}{G_{fib}}\end{cases}\qquad(20)$$

where $K_{\max}$ is the greater of K_{pm} and K_{fib}, $G_{fib} > G_{pm}$, $t = \left(\left\langle \tilde{G}^{-1} \right\rangle_{\eta} - 2(H/Z)\left\langle \tilde{K}^{-1} \right\rangle_{\zeta} \right) \Big/ \left(1 + (H/Z)\right)$,

$H = \sqrt{\eta_1 \eta_2 \left(G_{fib}^{-1} - G_{pm}^{-1}\right)^2}$, and $Z = \sqrt{\zeta_1 \zeta_2 \left(K_{fib}^{-1} - K_{pm}^{-1}\right)^2}$.

An estimate for G_e^T within these bounds is provided by Eq. (19), with d=2.

Transverse bulk modulus

The Silnutzer three-point lower bound on the transverse bulk modulus, K_e^T, is given by Eq. (14) and Eq. (15) (Silnutzer, 1972; Torquato, 2001) with, as above, ϕ_1 replaced with $(f_1 + f_2)$, ϕ_2 replaced with f_3, $\eta_2 = 0.276\, f_3$, $\zeta_2 = 0.691\, f_3 + 0.0428\, f_3^2$, and $\langle\ \rangle$ replaced with $\langle\ \rangle_f$.

A tighter upper bound is the Gibiansky-Torquato upper bound (Gibiansky and Torquato, 1995; Torquato, 2001):

$$K_e^T \le \langle K \rangle_f - \frac{(f_1 + f_2) f_3 \left(K_f - K_{pm}\right)^2 \langle G \rangle_{\zeta}}{\left\langle \tilde{K} \right\rangle_f + \dfrac{G_f G_{pm} + K_f}{K_f + \left\langle \tilde{G} \right\rangle_{\zeta}}} \tag{21}$$

An estimate for the lower bound K_e^T is obtained by Eq. (19), with d=2.

Longitudinal-transverse shear modulus

The shear modulus for distortion of the laminate in axes with one direction aligned with the fiber axes is given by the Hashin-Rosen (1964) bounds:

$$\frac{\langle G \rangle_f + G_{fib}}{\left\langle \tilde{G} \right\rangle_f + G_{pm}} G_{pm} \le G_e^{LT} \le \frac{\langle G \rangle_f + G_{pm}}{\left\langle \tilde{G} \right\rangle_f + G_{fib}} G_{fib}. \tag{22}$$

The Tsai-Halpin semi-empirical equations provide estimates within these bounds for values of the parameter ξ in the range $0 \le \xi \le 25$:

$$G_e^{LT} \approx \frac{1 + \xi \omega f_3}{1 - \omega f_3} G_{pm}, \tag{23}$$

where $\omega = \dfrac{\left(G_{fib} / G_{pm}\right) - 1}{\left(G_{fib} / G_{pm}\right) + \xi}$.

$$G_e^{LT} \approx \frac{G_{fib} + 2G_{pm} + 2\left(G_{fib} - G_{pm}\right) f_3}{G_{fib} + 2G_{pm} - \left(G_{fib} - G_{pm}\right) f_3} G_{pm}$$

$$\frac{dG_e^{LT}}{df_2} \sim -\left(\left(G_{fib} + 2G_{pm} + 2\left(G_{fib} - G_{pm}\right) f_3\right) G_{pm}\right)\left(2G'_{pm} + f_3 G'_{pm}\right)$$
$$+ \left(G_{fib} + 2G_{pm} - \left(G_{fib} - G_{pm}\right) f_3\right)\left(\left(2G'_{pm} - 2G'_{pm} f_3\right) G_{pm} + G'_{pm}\left(G_{fib} + 2G_{pm} + 2\left(G_{fib} - G_{pm}\right) f_3\right)\right)$$

$$\sim -G'_{pm}\left[\left(\left(G_{fib} + 2G_{pm} + 2\left(G_{fib} - G_{pm}\right) f_3\right) G_{pm}\right)\left(2 + f_3\right)\right.$$
$$\left. + \left(G_{fib} + 2G_{pm} - \left(G_{fib} - G_{pm}\right) f_3\right)\left(\left(2 - 2 f_3\right) G_{pm} + G_{fib} + 2G_{pm} + 2\left(G_{fib} - G_{pm}\right) f_3\right)\right]$$

$$\sim -G'_{pm}\left[\left(\left(G_{fib} + 2G_{pm} + 2\left(G_{fib} - G_{pm}\right) f_3\right) G_{pm}\right)\left(2 + f_3\right)\right.$$
$$\left. + \left(G_{fib} + 2G_{pm} - \left(G_{fib} - G_{pm}\right) f_3\right)\left(G_{fib} + 4G_{pm} + 2G_{fib} f_3 - 4G_{pm} f_3\right)\right]$$

The value $\xi = 2$ has been shown to provide a good estimate for laminae containing regularly spaced, aligned fibers, and will be adopted in the following.

Longitudinal stiffness

As described by Torquato (2001), the longitudinal stiffness E_e^L of a transversely isotropic two-phase lamina is dictated by the bounds or estimates of the aforementioned properties (Hill, 1964):

$$E_e^L = \langle E \rangle_f + \frac{4\left(\nu_{pm} - \nu_f\right)^2}{\left(\dfrac{1}{k_{fib}} - \dfrac{1}{k_{pm}}\right)^2}\left(\left\langle \frac{1}{k} \right\rangle_f - \frac{1}{K_e^T + G_e^T / 3}\right) \tag{24}$$

where ν is Poisson's ratio of the isotropic particular matrix (pm) or fibers (f), and $k_i = K_i + G_i / 3$ for each phase i.

Lateral/transverse Poisson's ratio

The effective Poisson's ratio ν_e^{LT} is dictated by the other material constants of the laminate (Hill, 1964; Torquato, 2001):

$$\nu_e^{LT} = \langle \nu \rangle_f + \frac{\left(\nu_{pm} - \nu_f\right)}{\left(\frac{1}{k_{pm}} - \frac{1}{k_{fib}}\right)} \left(\left\langle \frac{1}{k} \right\rangle_f - \frac{1}{K_e^T + G_e^T / 3} \right) \tag{24}$$

where ν is Poisson's ratio, and $k_i = K_i + G_i / 3$ for each phase i.

3.2 Moduli and bending stiffness of a 0/90 laminate composite

Three elastic constants are required to assemble the effective in-plane mechanical properties of a symmetric 0/90 laminate with equal numbers of laminae in the 0° and 90° directions. These can be written, for axes parallel to the fiber directions, as:

$$\begin{aligned} \nu_o &= \frac{2c_{12}}{c_{11} + c_{22} + 2c_{23}} \\ E_o &= \frac{c_{11} + c_{22}}{2} - 2c_{12}\nu_o \\ G_o &= G^{LT} \end{aligned} \tag{25}$$

where $c_{11} = \alpha E^L \left(1 - \nu_T^2\right)$, $c_{22} = \alpha E^T \left(1 - \frac{E^T}{E^L} \left(\nu^{LT}\right)^2\right)$, $c_{12} = \alpha E^T \nu^{LT} \left(1 + \nu^T\right)$, $c_{23} = \alpha E^T \left(\nu^T + \frac{E^T}{E^L} \left(\nu^{LT}\right)^2\right)$,

$\nu^T = \left(E^T / 2G^T\right) - 1$, and $\alpha = \left(1 - 2\left(\nu^T\right)^2 - \frac{E^T}{E^L}\left(\nu^{LT}\right)^2 \left(1 - 2\nu^T\right)\right)^{-1}$.

4. Effectiveness of Particulate-Matrix Fiber-Reinforced Composites

The following representative examples illustrate the advantages available adding a relatively small amount of spherical particles to the matrix of fiber-reinforced composites. Two cases considered here include simply supported panels subject to an instantaneously applied uniform pressure and the panels undergoing overpressure due to an explosive blast. We will focus only on choosing the volume fraction of spheres that optimizes weight and stiffness of the structure. Cross-ply panels considered here have dimensions a by b (in the x- and y-directions, respectively), thickness h, and density ρ. The mass m of such a panel is

$$m = \rho abt \tag{26}$$

The peak deflection δ due to an instantaneously applied uniform pressure, *p*, on one face of the panel, is obtained applying a magnification factor of $\mu = 2$ to the static one degree of freedom approximate solution that was shown to yield an error equal or smaller than 2.1% in isotropic panels (Lekhnitskii, 1968):

$$\delta \approx \mu\delta_{st} = \mu \frac{5pa^4}{384D_1}\left(1 + \frac{1}{k_1^2 - k_2^2}\right)\left(\frac{k_2^2}{\cosh(k_1 b/2a)} - \frac{k_1^2}{\cosh(k_2 b/2a)}\right) \tag{27}$$

where $k_{1,2} = \sqrt{9.871\frac{D_3}{D_1} \pm \sqrt{97.436\left(\frac{D_3}{D_1}\right)^2 - 97.548}}$, and the stiffnesses D_i are found in terms of plate bending stiffness as follows:

$$\begin{aligned}
&D_1 = D_{11}, \qquad D_2 = D_{22}, \qquad D_3 = D_{12} + 2D_{66} \\
&D_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij} z^3 dz \\
&\overline{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}s^4 \\
&\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(s^4 + c^4) \\
&\overline{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}c^4 \\
&\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4) \\
&s = \sin\theta, \qquad c = \cos\theta \\
&Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \qquad Q_{12} = \nu_{21}Q_{11}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \qquad Q_{66} = G_{12}
\end{aligned} \tag{28}$$

In (28), θ is the lamination angle of the layer, Q_{ij} are reduced stiffnesses, E_1 and E_2 are the layer moduli of elasticity in the fiber direction and in the direction perpendicular to fiber, respectively, G_{12} is the layer in-plane shear modulus, and ν_{12}, ν_{21} are the Poisson ratios.

In addition to the response to an instantaneously applied pressure, it is also possible to use the solution of Lekhnitskii shown above to predict the response of the panel to an explosive blast. Problems of response of structures to blast have been considered by a number of authors, including Houlston et al. (1985), Gupta (1985), Gupta et al. (1987), Birman and Bert (1987), Librescu and Nosier (1990), Librescu and Na (1998a, 1998b). The blast overpressure is usually uniformly distributed over the surface of the panel and can be presented by the Friedlander exponential decay equation:

$$p(t) = p\left(1 - \frac{t}{t_p}\right)\exp\left(-a't/t_p\right) \tag{29}$$

where p is a constant, t is time, t_p is a positive phase duration of the pulse and a' is an experimental decay parameter.

The response can now be evaluated using a single-degree of freedom Lekhnitskii's approximate solution combined with the convolution integral:

$$\delta(t) = \delta_{st} f(t) \tag{30}$$

where the function of time is obtained using integrals from the web site mathematica.com as follows (see Appendix 2 for a justification of the solution):

$$f(t) = \omega_n \int_0^t \left(1 - \frac{\xi}{t_p}\right)\exp\left(-\frac{a'\xi}{t_p}\right)\sin\left[\omega_n(t-\xi)\right]d\xi = \frac{\omega_n}{\left[(a')^2 + (\omega_n t_p)^2\right]^2}\left[I(t) - I(0)\right]$$

$$I(t) = \exp\left(-\frac{a't}{t_p}\right)(\omega_n t_p)\left[-2a't_p + \left((a')^2 - (\omega_n t_p)^2\right)(t_p - t)\right] \tag{31}$$

$$I(0) = t_p\left\{\omega_n t_p\left[-2a' + (a')^2 + (\omega_n t_p)^2\right]\cos\omega_n t + \left[(a')^2 + (\omega_n t_p)^2\right](1 - a')\sin\omega_n t\right\}$$

In (31), ω_n is the natural frequency. The latter frequency could be determined using (30) in the Ritz solution to the problem of free vibrations of the plate. However, considering the smallness of the error obtained using the static Lekhnitskii solution, it is possible to use the exact solution for the fundamental frequency of the plate, so that

$$\omega_n^2 = \frac{\pi^4}{\overline{m}}\left(\frac{D_1}{a^4} + \frac{2D_3}{a^2b^2} + \frac{D_2}{b^4}\right) \tag{32}$$

where $\overline{m}$ is the mass of the plate per unit surface area.

5. Numerical Results

Material constants of an epoxy matrix with spherical glass inclusions were evaluated using the Mori-Tanaka theory and compared to several available bounds in Fig. 1. The bounds included in this figure are those by Voight and Reuss, Hashin-Shtrikman, and three-point bounds. Additionally, the weak contrast estimate is provided in Fig. 1. Remarkably, the Mori-Tanaka estimate is always within the Voight-Reuss bounds. This estimate was also found close to the lower three-bound estimate by the Hashin-Shtrikman and by three-point bounds, for the elasticity, shear and bulk moduli, while the prediction for the Poisson ratio was close to the upper bounds according to these techniques. It should be noted that while the Mori-Tanaka predictions remain within the bounds according to Hashin-Shtrikman, they were slightly outside the bounds by the three-point technique, particularly at larger particle volume fractions, though the deviation remained quite small. Overall, the results shown in Fig. 1 support the previously cited opinion that the Mori-Tanaka theory is acceptable, at least for the particle volume fraction below 40%.

The results shown in Fig. 2 refer to the properties of a particulate-matrix fiber reinforced material (glass fibers and particles and epoxy matrix). The bounds shown in this figure rely on strict three-point bounds from Fig. 1 that are applied to the case of a fiber-reinforced composite according to the equations shown in the paper. All moduli are normalized by those of the epoxy matrix. As follows from the figure, the "Generalized Benveniste" estimate corresponding to Eq. (13) remains within the bounds only if both the fiber as well as the particle volume fractions are small. It is interesting to note that while the present theory predicts the longitudinal elastic modulus that is higher than the upper three-point bound, the transverse modulus is lower than the lower bound. This means that while the present model overestimates the longitudinal stiffness of a particulate-matrix fiber-reinforced lamina, the transverse stiffness is underestimated. Accordingly, the shortcomings of the model for a single lamina (that is never contemplated in realistic design, anyway), may cancel each other in a cross-ply material. This assumption is further investigated in Fig. 3. Remarkably, the result shown in this figure indicates that the predictions for the stiffness obtained by the present theory for a cross-ply material remain within the tightest three-point bounds (actually, almost coincide with the lower bound). This means that the theory can successfully be applied to the prediction of the stiffness of particulate-matrix fiber-reinforced cross-ply composites.

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Appendix: Eshelby's tensors (Zhao and Weng, 1990)

(1) Components of Eshelby's tensor for a spherical inclusion embedded within an isotropic matrix:

$$\begin{aligned} S_{1111} &= S_{2222} = S_{3333} = \frac{7-\nu_1}{15(1-\nu_1)} \\ S_{1122} &= S_{2233} = S_{3311} = \frac{5\nu_1-1}{15(1-\nu_1)} \\ S_{1212} &= S_{2323} = S_{3131} = \frac{4-5\nu_1}{15(1-\nu_1)} \end{aligned} \tag{A1}$$

where ν_1 is the matrix Poisson ratio.

(2) Eshelby's tensors for a cylindrical inclusion embedded within an isotropic matrix

$$\begin{aligned} S_{1111} &= S_{1122} = S_{1133} = 0 \\ S_{2222} &= S_{3333} = \frac{5-4\nu_1}{8(1-\nu_1)} \\ S_{2211} &= S_{3311} = \frac{\nu_1}{2(1-\nu_1)} \\ S_{2323} &= \frac{3-4\nu_1}{8(1-\nu_1)} \\ S_{1212} &= S_{1313} = 0.25 \end{aligned} \tag{A2}$$

Appendix 2: Response of the structure modeled by a single-degree-of-freedom system to an arbitrary dynamic load using the static solution

The dynamic response of a single degree of freedom undamped system to an arbitrary load is governed by the following equation of motion (all notations are standard for the theory of vibrations):

$$m\ddot{w} + kw = f(t) \tag{B1}$$

The convolution integral describing steady state vibrations is

$$w = \frac{1}{m\omega_n}\int_0^t f(\xi)\sin\omega_n(t-\xi)d\xi \tag{B2}$$

where ω_n is the natural frequency.

Consider a structural system modeled by a single-degree-of-freedom approximation as was done in the solution of Lekhnitskii for a rectangular simply supported plate. Let the equation of static equilibrium be

$$L(\delta_{st}) = \hat{p} \tag{B3}$$

$\hat{p}$ being a static load and δ_{st} the static solution that satisfies (B3).

If the same system is subject to a dynamic load $\hat{p}f(t)$, the equation of motion is

$$L(\delta) = \hat{p}f(t) - m\ddot{\delta} \tag{B4}$$

Assume the solution in the form $\delta = \delta_{st}F(t)$.

Then the substitution of (B3) into (B4) yields

$$m\delta_{st}\ddot{F} + L(\delta_{st})F = fL(\delta_{st}) \tag{B5}$$

From the comparison with (B1) and (B2) it is obvious that

$$F = \sqrt{\frac{L(\delta_{st})}{m\delta_{st}}}\int_0^t f(\xi)\sin\omega_n(t-\xi)d\xi = \omega_n\int_0^t f(\xi)\sin\omega_n(t-\xi)d\xi \tag{B6}$$

where the natural frequency is $\omega_n = \sqrt{\frac{L(\delta_{st})}{m\delta_{st}}}$.

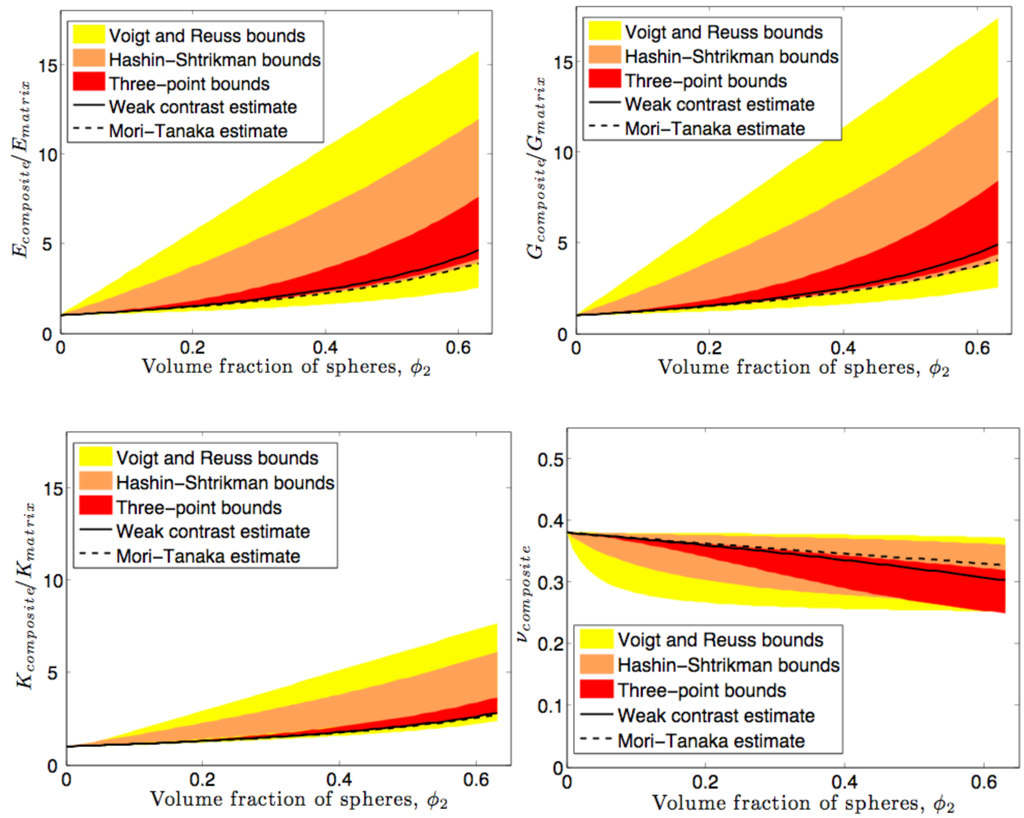

Figure 1. Estimates and bounds for the stiffening of an isotropic epoxy matrix by spherical glass inclusions.

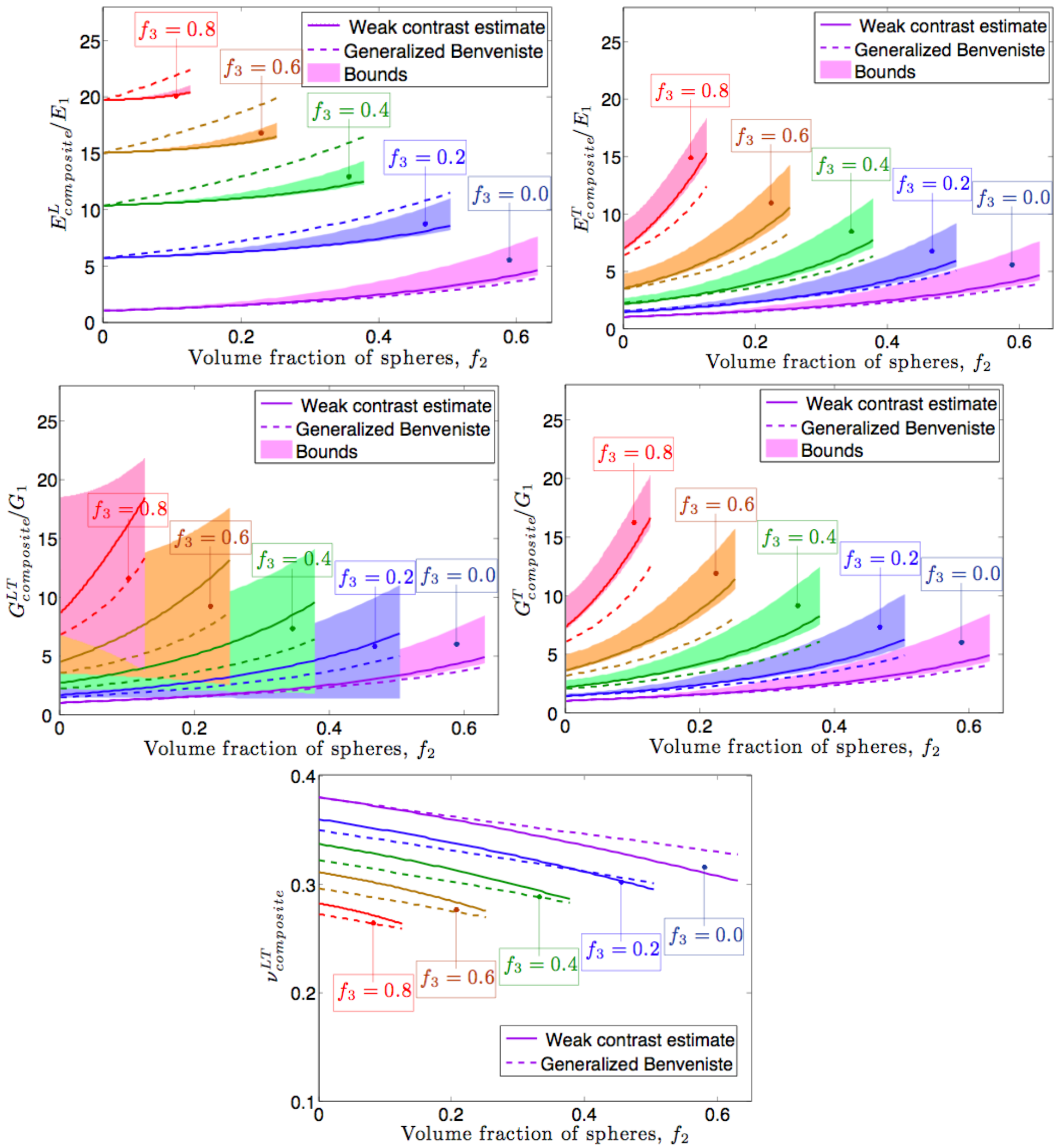

Figure 2. Estimates and bounds for the stiffening of a composite containing varying volume fractions f_3 of isotropic glass fibers by a volume fraction f_2 of spherical glass inclusions. The upper and lower bounds pictured are calculated using the upper and lower bounds from Figure 1 in conjunction with the bounding procedures described in the text. Moduli are normalized by those of epoxy.

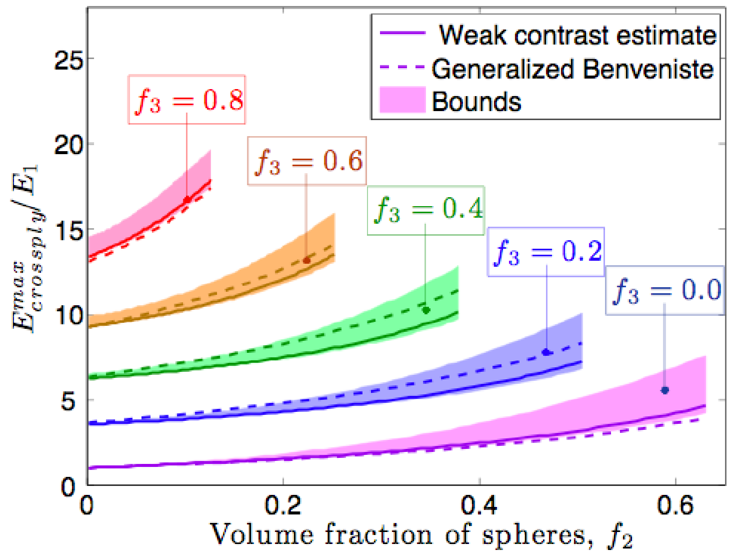

Figure 3. Estimates and bounds for the stiffening of a symmetric 0/90 cross-ply composite containing varying volume fractions f_3 of isotropic glass fibers by a volume fraction f_2 of spherical glass inclusions. Moduli are normalized by the elastic modulus of epoxy.